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Higher Order Polar Germs

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1. INTRODUCTION

Let $\xi : f = 0$, $f \in \mathbb{C}\{x, y\}$, be a germ of analytic curve at the origin of $\mathbb{C}^2$ and write n for its intersection multiplicity with the y-axis. In this note we will consider rth order polar germs of ξ of the form $\zeta^{r)} : \partial^r f/\partial y^r = 0$, $0 < r \leq n$. Ordinary (i.e., first order) polars have played an important role in both the classical and modern theory of singularities. By contrast, higher order polars seem not to have been considered from a local viewpoint, as they appear only in the classical projective theory of polar hypersurfaces (see, for instance, [5, Book III, Chap. I]). The fact that the topological type of the germ ξ does not determine the topological type of its first order polars [7] seems to indicate that not much can be said about the second and higher order polars and, indeed, higher order polar germs of many-branched germs have a rather erratic behaviour (see the examples in Section 5). Nevertheless, this is not the case if the germ ξ is unibranched: our main result here extends the decomposition theorem and related results of Merle [6] to the higher order polar germs $\zeta^{r)}$ of an irreducible germ of plane curve ξ. As a consequence we reprove a result of Dickenstein and Sessa [4], namely that the intersection multiplicities $[\xi.\zeta^{r)}]$, $0 < r < n$, determine and are in turn determined by the characteristic exponents of the Puiseux series of ξ (or by the topological type of ξ if the y-axis is taken non-tangent to ξ).

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2. PRELIMINARIES

We will consider germs of analytic curves traced on analytic smooth complex surfaces; they will be called germs of curve or just germs. Once local coordinates x, y at a point O on a smooth surface are fixed, we write $\xi : f = 0$ to indicate that the germ ξ at O is defined by the equation $f \in \mathbb{C}\{x, y\}$. Germs $y = 0$ and $x = 0$ will be called, respectively, the first and second axis. We use additive notation for germs so that if $\xi_i : f_i = 0$, $i = 1, 2$, then the germ composed of ξ_1 and ξ_2, $f_1 f_2 = 0$, is written $\xi_1 + \xi_2$. As usual, irreducible components of germs are called branches.

If g is any germ of function defining a smooth germ $\eta : g = 0$ with origin at O and ξ is a germ with origin at O, any germ

$$P_g(f) : \frac{\partial(f, g)}{\partial(x, y)} = 0,$$

f an equation of ξ, is called a (first order) g-polar germ of ξ. It is easy to see that the definition depends on the equation f but not on the local coordinates. If these are taken with $x = g$, then $P_g(f) : \partial f / \partial y = 0$. The equation of the polar identically vanishes (and therefore the polar germ remains undefined) if and only if η is the only irreducible component of ξ. In the sequel we will exclude this case and also the case in which η is a component of ξ, as it has little interest.

Write $n = [\xi . \eta]$, where square brackets denote intersection multiplicity. It is easy to check that $[P_g(f) . \eta] = n - 1$. In particular if η is not a component of ξ, it is not a component of $P_g(f)$ either, which allows us to define the rth order polars of ξ inductively, by taking $P_g^{1)}(f) = P_g(f)$ and $P_g^{r)}(f) = P_g(P_g^{r-1)}(f))$ for $r = 2, \ldots, n$. If the local coordinates x, y are chosen so that $x = g$, then the equations of the rth g-polars are just the rth y-derivatives, namely $P_g^{r)}(f) : \partial^r f / \partial y^r = 0$. We will often write $P_g^{r)}(\xi)$ for $P_g^{r)}(f)$ if no explicit mention of the equation f of ξ is required.

Next we briefly recall some classical facts about the Newton polygons and Puiseux series of germs; they will be used in the sequel without further reference. For more details the reader is referred, among many others, to [1, 3, 8]. Let $f = \sum_{\alpha, \beta \geq 0} a_{\alpha, \beta} x^{\alpha} y^{\beta} \in \mathbb{C}\{x, y\}$. The Newton diagram of f is $\Delta(f) = \{(\alpha, \beta) \mid a_{\alpha, \beta} \neq 0\} \subset \mathbb{R}^2$. The border of the convex envelope of $\Delta(f) + (\mathbb{R}^+)^2$ is composed of two half-lines parallel to the axis and a polygonal line (maybe reduced to a point) joining them which is called the Newton polygon of f, $\mathbf{N}(f)$. Assume that $\mathbf{N}(f)$ has vertices (α_i, β_i), $\beta_{i-1} > \beta_i$, $i = 1, \ldots, d$. If Γ is the side of $\mathbf{N}(f)$ with ends $(\alpha_{i-1}, \beta_{i-1})$,

(α_i, β_i), the polynomial

$$\Omega_\Gamma = \sum_{(\alpha,\beta)\in\Gamma} a_{\alpha,\beta} z^{\beta-\beta_i}$$

is called the equation associated to Γ. If the slope of Γ is, in irreducible form, $-n'/m'$, then Ω_Γ is a polynomial in $z^{n'}$ and hence its roots are distributed in conjugacy classes $\bar{a} = \{\varepsilon a\}_{\varepsilon^{n'}=1}$.

Clearly the first (resp. second) axis is a branch of $f = 0$ if and only if $\beta_d > 0$ (resp. $\alpha_0 > 0$) and in such a case β_d (resp. α_0) is its multiplicity as branch of $f = 0$. Note that the second axis has no Puiseux series, while the Puiseux series of the first axis is 0.

If γ is a branch of the germ $\xi : f = 0$ different from either axis, then the initial terms of the Puiseux series of γ are $ax^{m'/n'}$, where $-n'/m'$ is the slope of a side Γ of $\mathbf{N}(f)$ and a describes a conjugacy class of roots of Ω_Γ. Conversely, for any side Γ of $\mathbf{N}(f)$ and any conjugacy class $\bar{a}$ of roots of Ω_Γ, there is at least one branch γ of ξ whose Puiseux series have their initial terms as above. If ζ is the germ composed of all such branches γ (branches being counted according to their multiplicities in ξ), its intersection multiplicity with the second axis $\eta : x = 0$ is $[\zeta.\eta] = \mu n'$, where μ is the multiplicity of a as a root of Ω_Γ and n' the irreducible numerator of the opposite of the slope of Γ. As a consequence the intersection multiplicity with the second axis of the germ composed of all branches corresponding to the side Γ equals the degree of Ω_Γ, that is, the height $\beta_{i-1} - \beta_i$ of Γ.

We have found no reference for the following elementary result that will be useful in the sequel:

LEMMA 2.1. *The rth derivative of $F = (z^n - 1)^l$ shares no root with F if $l \le r \le nl$.*

Proof. An easy induction shows that

$$\frac{d^r F}{dz^r} = \sum_{i=0}^{l} c_i^r z^{n(l-i)-r} (z^n - 1)^i,$$

where the c_i^r are non-negative integers. Since 1 is an lth root of F, $c_0^l \neq 0$, after which the monomial $c_0^l z^{ln-l}$ gives rise, by derivation, to non-zero monomials in all further derivatives $d^r F/dz^r$ for $r = l+1, \dots, ln$. These monomials cannot be cancelled because no c_i^r is negative. Thus $c_0^r \neq 0$ for $r = l \dots ln$ and the claim follows. ∎

3. DECOMPOSITION THEOREM

Let $\xi : f = 0$ be an irreducible germ of curve with origin at a point O. Assume that an arbitrary system of local coordinates x, y at O has been chosen with the only condition of having its second axis, namely $\eta : x = 0$, different from ξ. Assume that a Puiseux series of ξ relative to the coordinates x, y is

$$s = \sum_{i \geq 1} a_i x^{i/n}.$$

Its characteristic exponents will be written $m_1/n, \dots, m_k/n$; we put $n = n_0$ and $n_i = \gcd\{n, m_1, \dots, m_i\}$, $1 \leq i \leq k$, and assume that the common denominator n has been chosen so that $n_k = 1$. In this case n is called the polydromy order of s; it equals the number of different Puiseux series of ξ. These are

$$s_\varepsilon = \sum_{i \geq 1} \varepsilon^i a_i x^{i/n},$$

for $\varepsilon^n = 1$, and $n = [\xi.\eta]$. In the sequel we will deal with x-polars $P_x^{r)}(\xi)$, $1 \leq r \leq n$.

We will refer to the case in which the second axis η is not tangent to ξ as the *transverse case*, the x-polars $P_x^{r)}(\xi)$ then being called *transverse polars*. As it is well known, in the transverse case the characteristic exponents of s do not depend on the coordinates and determine and are in turn determined by the topological or equisingularity type of ξ: they are then called the *characteristic exponents* of ξ. In the non-transverse case, by the inversion formula, the characteristic exponents of s determine and are determined by the characteristic exponents of ξ together with the intersection multiplicity n of ξ and the second axis (see [3, 11]). Thus in any case the characteristic exponents of s depend only on the germ ξ and the first coordinate function x, and not on y. In the present section and in the next one we will show that the characteristic exponents of s and the x-polars of ξ are closely related. We will deal with both the transverse and the non-transverse cases. The first one being the most significant, the latter has its own interest and is needed for induction purposes.

Keeping the conventions and hypotheses introduced above, our main result reads:

THEOREM 3.1. *Let $r \in \mathbb{Z}$, $1 \leq r < n$, and take $u(r)$ so that $n_{u(r)-1} > r \geq n_{u(r)}$. Any rth x-polar germ of ξ decomposes*

$$P_x^{r)}(\xi) = \zeta_1^{r)} + \cdots + \zeta_{u(r)}^{r)},$$

where

(a)

$$\left[\zeta_i^{r)}.\eta\right] = r\left(\frac{n}{n_i} - \frac{n}{n_{i-1}}\right) \qquad \text{if } i < u(r), \qquad \left[\zeta_{u(r)}^{r)}.\eta\right] = n - \frac{rn}{n_{u(r)-1}};$$

(b) *for each* $i = 1, \dots, u(r)$, *all branches of* $\zeta_i^{r)}$ *have Puiseux series of the form*

$$s_i^{r)} = \sum_{1 \le j < m_i} a_j x^{j/n} + c(x^{\alpha} + \cdots),$$

where either $c = 0$ *or, otherwise, dots indicate terms of higher order,* $\alpha \geq m_i/n$, *and in case of equality* $c^{n_{i-1}/n_i} \neq a_{m_i}^{n_{i-1}/n_i}$.

Remark 3.2. Part (a) clearly assures that no germ $\zeta_i^{r)}$ is empty.

Remark 3.3. As is easy to check, if η is not tangent to ξ, it is not tangent to any $P_x^{r)}(\xi)$ either. Therefore, in the transverse case part (a) of the claim gives the multiplicities of the components $\zeta_i^{r)}$.

Remark 3.4. Part (b) says that all branches of $\zeta_i^{r)}$ have a Puiseux series with partial sum

$$\sum_{1 \le j < m_i} a_j x^{j/n}$$

and no Puiseux series of a branch of $\zeta_i^{r)}$ shares with s a partial sum of higher degree. In either form, part (b) obviously implies that $\zeta_i^{r)}$ and $\zeta_{i'}^{r)}$ share no branch if $i \neq i'$.

Remark 3.5. As follows from Enriques' theorem (see [5, Book IV, Chap. I; 1, Chap. 8; 3, Chap. 5]), the infinitely near points on ξ that either belong to the second axis or are satellite points are distributed in k clusters of consecutive points, each cluster corresponding to a characteristic exponent of s. Because of the way the terms of s determine the infinitely near points on ξ (again by Enriques' theorem), part (b) of 3.1 says that all branches of $\zeta_i^{r)}$ go through all points preceding the ith cluster while no branch goes through a point after it.

Proof of Theorem 3.1. The claim for $n = 1$ (i.e., ξ smooth and non-tangent to η) being empty, we assume $n > 1$ and hence $k \geq 1$. It is enough to make the proof taking $a_{m_1}^{-1}(y - \sum_{j<m_1} a_j x^{j/n})$ instead of y, as a new second coordinate. Indeed, this clearly does not affect claim (b) and neither η nor the x-polars nor the characteristic exponents of s depend on

the second coordinate y. Thus we assume in the sequel that $a_j = 0$ for $j < m_1$ and $a_{m_1} = 1$. If f is any equation of ξ, up to a non-zero constant factor it has the form

$$f = (y^{n'} - x^{m'})^{n_1} + \sum_{n\alpha + m_1\beta > nm_1} a_{\alpha,\beta} x^\alpha y^\beta,$$

where $n' = n/n_1$, $m' = m_1/n_1$. Therefore $\mathbf{N}(f)$ has a single side Γ, with ends $(0, n)$ and $(m_1, 0)$ and whose integral points are $P_t = (tm', n - tn')$, $t = 0, \dots, n_1$, each corresponding to a non-zero coefficient of f. Write $\Omega = (z^{n'} - 1)^{n_1}$, the equation associated to Γ. Clearly $\Delta(\partial^r f/\partial y^r)$ comes from $\Delta(f)$ by deleting all points (α, β) with $\beta < r$ and moving the remaining points r steps downward. Thus $\Delta(\partial^r f/\partial y^r)$ has no points below the line parallel to Γ through $(0, n - r)$ and its points on this line are $\overline{P}_t = (tm', n - tn' - r)$, $t \leq (n - r)/n'$. Write $l = [(n - r)/n']$; $\mathbf{N}(\partial^r f/\partial y^r)$ thus has its upper end at $\overline{P}_0$, a side $\overline{\Gamma}$ from $\overline{P}_0$ to $\overline{P}_l$ only in case of $l > 0$, and all remaining sides with slope strictly bigger than $-n/m_1$. Furthermore, in case it does exist, $\overline{\Gamma}$ has associated equation $\overline{\Omega} = z^{-n+ln'+r} d^r\Omega/dz^r$. We define the component $\zeta_1^{r)}$ of $P_x^{r)}(f)$ as composed of all branches of $P_x^{r)}(f)$ but those (if any) corresponding to the side $\overline{\Gamma}$ and the conjugacy class of the root 1 of $\overline{\Omega}$ (see Section 2), branches taken according to their multiplicities in $P_x^{r)}(f)$. As is clear (and already known) the y-axis is not a branch of $P_x^{r)}(f)$. Hence, by its own definition, a branch of $\zeta_1^{r)}$ either corresponds to the side $\overline{\Gamma}$ and a conjugacy class of roots of $\overline{\Omega}$ other than $\overline{1}$, or it corresponds to a further side of $\mathbf{N}(\partial^r f/\partial y^r)$, which therefore has slope bigger than $-n/m_1$, or it equals the x-axis. In any case it has a Puiseux series as claimed in (b), which is thus proved for $i = 1$.

We consider first the case $r \geq n_1$, which forces $u(r) = 1$. Then either $\overline{\Gamma}$ does not exist, or otherwise $\overline{\Omega}$ has no root 1, by 2.1. Thus in this case $P_x^{r)}(f) = \zeta_1^{r)}$ and clearly condition (a) is satisfied. Thus, if $r \geq n_1$, the proof is complete. Note that this is the case if $k = 1$, so the proof will be continued using induction on the number k of characteristic exponents.

Assume thus that $r < n_1$: now $u(r) > 1$. $\overline{\Gamma}$ does exist, and $\overline{\Omega}$ has the root 1 with multiplicity $n_1 - r$. Put $P_x^{r)}(f) = \zeta_1^{r)} + \zeta'$. By the definition of $\zeta_1^{r)}$, ζ' is composed of all branches of $P_x^{r)}(f)$ corresponding to the side $\overline{\Gamma}$ and the conjugacy class of 1; hence $[\zeta'.\eta] = n'(n_1 - r)$ and therefore $[\zeta_1^{r)}.\eta] = n - r - n'(n_1 - r) = n'r - r$ as claimed in (a). Thus it remains to decompose ζ' in components $\zeta_i^{r)}$, $i = 2, \dots, u(r)$, satisfying claims (a) and (b), which we will do next.

Let p be the first infinitely near point on ξ after those depending on the characteristic exponent m_1/n. The point p lies on a smooth surface $\overline{S}$ related to S by the composition of the blowing-ups giving rise to p,

$\pi : \bar{S} \to S$. We will write E for the germ at p of the exceptional divisor of π. For any germ ζ with origin at O, $\pi^*(\zeta)$ and $\tilde{\zeta}$ will denote, respectively, the germs at p total transform (pull-back) and strict transform of ζ, the latter being determined by the conditions of having no component E and being $\pi^*(\zeta) = \rho E + \tilde{\zeta}$, $\rho \geq 0$.

By [2, 10.2], there are local coordinates $\tilde{x}, \tilde{y}$ on S at p related to the pull-backs $\bar{x} = \pi^*(x)$, $\bar{y} = \pi^*(y)$ of x, y by the formulas

$$\bar{x} = \tilde{x}^{n'}$$
$$\bar{y} = \tilde{x}^{m'}(1 + \tilde{y})$$

and such that E has equation $\tilde{x} = 0$. An easy computation shows that an irreducible germ at O, γ goes through p if and only if it has a Puiseux series of the form

$$s' = x^{m'/n'} + \sum_{j > dm'} b_j x^{j/dn'}.$$

In such a case its strict transform with origin at p, $\tilde{\gamma}$ has Puiseux series

$$\tilde{s}' = \sum_{j > dm'} b_j \tilde{x}^{(j - dm')/d},$$

and the correspondence $s' \mapsto \tilde{s}'$ is one to one between the set of Puiseux series of γ with initial coefficient equal to 1 and the whole set of Puiseux series of $\tilde{\gamma}$. This applies in particular to ξ and shows its strict transform $\tilde{\xi}$ to have a Puiseux series

$$\tilde{s} = \sum_{j > m_1} a_j \tilde{x}^{(j - m_1)/n_1},$$

with characteristic exponents $\overline{m}_i/\overline{n} = (m_{i+1} - m_1)/n_1$, $i = 1, \dots, k-1$. We take $\overline{n} = n_1$, after which $\overline{n}_i = \gcd(\overline{n}, \overline{m}_1, \dots, \overline{m}_1) = n_{i+1}$ and so, in particular, $\overline{n}_{k-1} = 1$. Now $\overline{n}_{u(r)-2} > r \geq \overline{n}_{u(r)-1}$; hence we put $\overline{u}(r) = u(r) - 1$. Notice that the induction hypothesis applies to $\tilde{\xi}$, as its Puiseux series has $k - 1$ characteristic exponents and $r < n_1 = \overline{n}$.

Let us have a look at the strict transform of $P_x^{r)}(\xi)$. Direct computation from the equation f using the formulas above shows that $\pi^*(f) = \tilde{x}^{nm'}\tilde{f}$, where $\tilde{f} \in \mathbb{C}\{\tilde{x}, \tilde{y}\}$, has no factor $\tilde{x}$ and hence is an equation of $\tilde{\xi}$. On one hand,

$$\frac{\partial^r \pi^*(f)}{\partial \tilde{y}^r} = \tilde{x}^{nm'} \frac{\partial^r \tilde{f}}{\partial \tilde{y}^r},$$

while on the other

$$\frac{\partial^r \pi^*(f)}{\partial \tilde{y}^r} = \frac{\partial^r f(\bar{x}, \bar{y})}{\partial \tilde{y}^r} = \tilde{x}^{rm'} \pi^* \left(\frac{\partial^r f}{\partial y^r} \right).$$

Both together give

$$\pi^*\left(\frac{\partial^r f}{\partial y^r}\right) = \tilde{x}^{(n-r)m'}\frac{\partial^r \tilde{f}}{\partial \tilde{y}^r}.$$

Since $r < \bar{n} - [\tilde{\xi}.E]$, $\partial^r f/\partial \tilde{y}^r$ has no factor $\tilde{x}$ and so we have seen that the strict transform of our rth polar of ξ is an rth polar of $\tilde{\xi}$; more precisely,

$$\left(\widetilde{P_x^{r)}(f)}\right) = P_{\tilde{x}}^{r)}(\tilde{f}).$$

By their own definitions, any branch of $\zeta_1^{r)}$ does not go through p while all branches of ζ' do. Thus not only does

$$\tilde{\zeta}' = \left(\widetilde{P_x^{r)}(f)}\right) = P_{\tilde{x}}^{r)}(\tilde{f}),$$

but also taking strict transforms gives a one to one correspondence between the branches of ζ' and those of $P_{\tilde{x}}^{r)}(\tilde{f})$, this correspondence preserving the multiplicities of the branches as irreducible components of the germs. In this situation we are allowed to translate to ζ' the decomposition of $P_{\tilde{x}}^{r)}(\tilde{f})$ provided by the induction hypothesis. Indeed, assume that $P_{\tilde{x}}^{r)}(\tilde{f}) = \bar{\zeta}_1^{r)} + \cdots + \bar{\zeta}_{\bar{u}(r)}^{r)}$ in the conditions of the claim. For each $i = 2, \dots, u(r) = \bar{u}(r) + 1$, assume that $\bar{\zeta}_{i-1}^{r)} = (\tilde{\gamma}_1) + \cdots + (\tilde{\gamma}_h)$, each γ_t a branch of ζ', and then take $\zeta_i^{r)} = \gamma_1 + \cdots + \gamma_h$. After this clearly $\zeta' = \zeta_2^{r)} + \cdots + \zeta_{u(r)}^{r)}$ and therefore $P_x^{r)}(\xi) = \zeta_1^{r)} + \cdots + \zeta_{u(r)}^{r)}$.

Fix $i, 2 \le i \le u(r)$. For any irreducible germ γ at O going through p, the equality $\bar{x} = \tilde{x}^{n'}$ directly gives $[\gamma.\eta] = [\tilde{\gamma}.\pi^*(\eta)] = n'[\tilde{\gamma}.E]$. Adding up for all branches of $\zeta_i^{r)}$, $[\zeta_i^{r)}.\eta] = n'[\bar{\zeta}_{i-1}^{r)}.E]$ and so the induction hypothesis gives

$$\left[\zeta_i^{r)}.\eta\right] = n'r\left(\frac{\bar{n}}{\bar{n}_{i-1}} - \frac{\bar{n}}{\bar{n}_{i-2}}\right) = r\left(\frac{n}{n_i} - \frac{n}{n_{i-1}}\right)$$

if $i < u(r)$, and

$$\left[\zeta_{u(r)}^{r)}.\eta\right] = n'\left(\bar{n} - \frac{r\bar{n}}{\bar{n}_{\bar{u}(r)-1}}\right) = \left(n - \frac{rn}{n_{u(r)-1}}\right).$$

as claimed in (a).

Lastly, for claim (b), assume that γ is any branch of $\zeta_i^{r)}$. By induction its strict transform has a Puiseux series

$$\sum_{1 \le j < \bar{m}_{i-1}} a_{j+m_1}\tilde{x}^{j/\bar{n}} + c(\tilde{x}^{\bar{\alpha}} + \cdots),$$

where if $c \neq 0$, then $\bar{\alpha} \geq \bar{m}_{i-1}/\bar{n}$ and in case of equality $c^{\bar{n}_{i-2}/\bar{n}_{i-1}} \neq a_{\bar{m}_{i-1}+m_1}^{\bar{n}_{i-2}/\bar{n}_{i-1}}$. It follows from the relationship between the Puiseux series of γ and those of its strict transform, explained above, that γ then has a Puiseux series

$$x^{m_1/n} + \sum_{1 \leq j < m_i - m_1} a_{j+m_1} x^{(j+m_1)/n} + c\left(x^{(n_1\bar{\alpha}+m_1)/n} + \cdots\right)$$

which satisfies the conditions of the claim. ▮

In fact, the above proof gives some further information on the branches of $\zeta_i^{r)}$, namely:

COROLLARY 3.6 (of the proof of 3.1). *The germ composed of the branches of $\zeta_i^{r)}$ that have ith characteristic exponent m_i/n (counted according to multiplicities) has intersection with η equal to*

$$\frac{n}{n_i}\left[\left(1 - \frac{n_i}{n_{i-1}}\right)r\right],$$

with the square brackets meaning the integral part. In particular there are no such branches if either $r = 1$ or $i = k$.

Proof. Assume first $i = 1$. The branches of $\zeta_1^{r)}$ whose Puiseux series have an initial term of degree strictly bigger than m_1/n correspond to the sides of the Newton polygon of $P_x^{r)}(\xi)$ other than $\bar{\Gamma}$. Thus they compose a germ whose intersection multiplicity with the second axis is the ordinate of the last end $\bar{P}_l$ of $\bar{\Gamma}$, namely $n - ln' - r$. Since the intersection of the whole $\zeta_1^{r)}$ with the second axis has been computed in 3.1, the claim for $i = 1$ follows.

Now, if $i > 1$ and therefore $r < n_1$, the same inductive argument used in the proof of 3.1 completes this one. ▮

4. PLÜCKER'S FUNCTION OF AN IRREDUCIBLE GERM

Hypotheses and notations being as in Theorem 3.1, next we show some consequences of it.

COROLLARY 4.1. *If γ is a branch of $\zeta_i^{r)}$, then*

$$[\xi.\gamma] = [\gamma.\eta]\left(\frac{(n-n_1)m_1 + \cdots + (n_{i-2} - n_{i-1})m^{i-1} + n_{i-1}m_i}{n}\right). \quad (1)$$

Proof. Let $s' = s_i^{r)}$ be the Puiseux series of γ in 3.1(b) and call ρ the polydromy order of s'. Then

$$[\xi.\gamma] = \operatorname{ord}_x \prod_{\substack{\varepsilon^n=1\\ \tau^\rho=1}} (s_\varepsilon - s'_\tau) = \rho \operatorname{ord}_x \prod_{\varepsilon^n=1} (s_\varepsilon - s'),$$

after which the claim follows from 3.1(b), by direct computation, as $\rho = [\gamma.\eta]$. ∎

If η is not tangent to ξ the quotients $[\xi.\gamma]/[\gamma.\eta]$, for γ a branch of $P_x^1(\xi)$, are called the *polar quotients* or the *polar invariants* of ξ (Teissier [10]; note that $[\gamma.\eta]$ is the multiplicity of γ, as in the transverse case no branch of the polar is tangent to η). For $r = 1$ the formula (1), which computes the polar invariants from the characteristic exponents of ξ, was proved by Merle [6].

The reader may notice that the right side of (1) does not depend on r. Hence, if the polar invariants are ordered according to their values, using 3.2, we have

COROLLARY 4.2. *If η is not tangent to ξ, for any branch γ of $\zeta_i^{r)}$ the quotient $[\xi.\gamma]/[\gamma.\eta]$ equals the ith polar invariant of ξ. If γ describes all branches of $P_x^{r)}(\xi)$, then these quotients describe the set of the first $u(r)$ polar invariants of ξ.*

After adding up the equalities (1) of 4.1 for all branches of $\zeta_i^{r)}$ (branches counted according to multiplicities) and using 3.1(a), we get

$$\left[\xi.\zeta_i^{r)}\right] = r\left(\frac{n}{n_i} - \frac{n}{n_{i-1}}\right) \times \left(\frac{(n-n_1)m_1 + \cdots + (n_{i-2}-n_{i-1})m^{i-1} + n_{i-1}m_i}{n}\right)$$

for $i < u(r)$, and

$$\left[\xi, \zeta_{u(r)}^{r)}\right] = \left(n - \frac{rn}{n_{u-1}}\right) \times \left(\frac{(n-n_1)m_1 + \cdots + (n_{u-2}-n_{u-1})m^{u-1} + n_{u-1}m_u}{n}\right),$$

where $u = u(r)$. It is then enough to add up once again, for $i = 1, \dots, u(r)$, to get a formula due to Dickenstein and Sessa [4]:

COROLLARY 4.3. *Let r be such that $n > r \geq 1$ and assume that $n_{u-1} > r \geq n_u$. If $P_x^{r)}(\xi)$ is any rth x-polar of ξ, then*

$$\left[\xi.P_x^{r)}(\xi)\right] = nm_1 + n_1(m_2 - m_1) + \cdots + n_{u-1}(m_u - m_{u-1}) - rm_u.$$

Dickenstein and Sessa's proof of 4.3 is by a nice direct computation. The case $r = 1$ is classical, as intersection multiplicities $[\xi.P_x^{1)}(\xi))]$, ξ a non-necessarily irreducible germ, are computed in order to establish the Plücker formula for the class of an algebraic curve (see [5, Book IV, II.16], for instance; see also [9, 5.5.5]). Since $P_x^{n)}(\xi)$ is an empty germ, we may add to the equalities of 4.3 the obvious one $[\xi.P_x^{n)}(\xi)] = 0$. As a direct consequence we have:

THEOREM 4.4 [4]. *The intersection multiplicities* $[\xi.P_x^{i)}(\xi)]$, $0 < i < n$, *determine and are in turn determined by the characteristic exponents of the Puiseux series of* ξ *relative to any local coordinates* x, y (*or just the characteristic exponents of* ξ *if* $\eta : x = 0$ *is not tangent to* ξ).

Proof. Directly from 4.3, $m_1 = [\xi.P_x^{n-1)}(\xi)]$. If we inductively assume that $m_1, \dots, m_i$ are determined, so is n_i and in case $n_i > 1$, again from 4.3, $m_{i+1} = [\xi.P_x^{n_i - 1)}(\xi)] - [\xi.P_x^{n_i)}(\xi)]$. ∎

We may call the *Plücker function* of the germ ξ relative to the function x the map $\mathscr{P}_x^{\xi}(r) = [\xi.P_x^{r)}(\xi)]$ for $r = 1, \dots, n$. By 4.3, $\mathscr{P}_x^{\xi}$ does not depend on which rth x-polars are used to define it. As the reader may easily check, the equalities on 4.3 and the further one $\mathscr{P}_x^{\xi}(n) = 0$ fit together to extend $\mathscr{P}_x^{\xi}$ to a continuous piecewise linear function defined in the closed interval $[1, n]$, this function being linear of slope $-m^i$ in $[n_i, n_{i-1}]$. In the transverse case ($\eta : x = 0$ not tangent to ξ) the Plücker function does not depend on x but only on the characteristic exponents of ξ and does in turn determine them.

5. A NOTE ON NON-IRREDUCIBLE GERMS

Of course, in case the germ ξ is no longer assumed to be irreducible it still makes sense to intersect ξ with branches of its higher order polars or with the polars themselves. Nevertheless, even for reduced germs and in the transverse case, such intersection multiplicities need not to be finite, nor are they determined by the topological type of ξ. In particular 4.3 cannot be extended to non-irreducible germs. Next are two examples:

EXAMPLE 5.1. Take $\xi : f = y^3 + x^2 y = 0$. Its second order polar $P_x^{2)}(f)$ has $[\xi.P_x^{2)}(f)] = \infty$.

EXAMPLE 5.2. Take $\xi_a : f_a = y^4 + ax^2y^2 + x^2y + x^{10} = 0$. Any germ ξ_a is composed of two branches: one is smooth and tangent to the first axis while the Puiseux series of the other has single characteristic exponent $2/3$. Neither the topological type of ξ_a nor the intersection multiplicities of its branches with the second axis thus depend on a. Nevertheless, $[\xi_a.P_x^{2)}(f_a)] = 6$ if $a \neq 0$ while $[\xi_0.P_x^{2)}(f_0)] = 20$.

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